A decomposition theorem in K_{ex} (Model Theory and It's Application to Algebra)

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A decomposition theorem in K_{ex}

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Abstract

Let S be a finite subset of $\mathbb{Q}[x_1, \cdots, x_n, y_1, \cdots, y_n]$ and $E = \{y_1 - e^{x_1}, \cdots, y_n - e^{x_n}\}$. A theorem of Richardson gives an irreducible decomposition of the set defined as the zero set of S and E. The same statement holds on K_{ex}, an algebraically closed field of characteristic zero with pseudo-exponentiation.

1 Decomposition of exponential-algebraic sets

1.1 Richardson's Theorem in $\mathbb{C}_{ex}$

In this sub section, we work in $\mathbb{C}_{\mathrm{exp}}$ the field of complex numbers with the complex expoential.

Let $\bar{x} = x_1, \cdots, x_n$ and $\bar{y} = y_1, \cdots, y_n$. Suppose $S = \{p_1(\bar{x}, \bar{y}), \cdots, p_k(\bar{x}, \bar{y})\}$ and $E = \{y_1 - e^{x_1}, \cdots, y_n - e^{x_n}\}$. Put $\mathbb{C}^{2n}(S, E) = \{(x, y) \mid p_i(x, y) = 0, y_j - e^{x_j} = 0,\ (i = 1, \cdots, k, j = 1, \cdots, n)\}$.

In [Ri], Richardson proves the following theorem.

Theorem 1 (Richardson) *For any finite set $S \subset \mathbb{Q}[\bar{x}, \bar{y}]$, the zero set of exponential system (S, E) can be written as a union of finitely many sets;*

$$\mathbb{C}^{2n}(S, E) = \bigcup_i C_i$$

where each set C_i is the zero set of a triangular condition Δ_i.

To make the statement precise we need some definitions.

First introduce a term-order among variables, e.g., $x_1 \prec x_2 \prec \cdots \prec x_n$. With this order we introduce a well-founded order on the polynomials in $\mathbb{Q}[x_1, \cdots, x_n]$.

Let S be a finite set of polynomials $\{p_1(\bar{x}, \bar{y}), \cdots, p_k(\bar{x}, \bar{y})\}$. If $p_1 \prec \cdots \prec p_k$, we say that S is an *ascending* set.

To define the notion of triangular condition we need the following definition of a *differential matrix*.

Definition 2 *Let S be a finite subset of $\mathbb{Q}[x, y]$ and $E_1 = \{y_{i_0} - e^{x_{i_0}}, \cdots, y_{i_l} - e^{x_{i_l}}\} \subseteq E$.*

$$df(S, E_1) = \begin{pmatrix} \frac{\partial p_1}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial p_1}{\partial x_n} & \frac{\partial p_1}{\partial y_1} & \cdots & \cdots & \cdots & \frac{\partial p_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial p_k}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial p_k}{\partial x_n} & \frac{\partial p_k}{\partial y_1} & \cdots & \cdots & \cdots & \frac{\partial p_k}{\partial y_n} \\ 0 & -y_{i_0} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -y_{i_l} & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Definition 3 (Triangular conditions) (S, E_1) is a triangular condition, if

1. S is an ascending set.
2. Let J be the determinant of a maximal minor of the differential matrix $df(SmE_1)$. Noticethat J is a polynomial. We infer that $Rem(J, S) \neq 0$.
3. Let $E_2 = E - E_1$. Then E_2 is such that if $C(t)$ is any smooth curve in $\mathbb{C}^{2n}$ which $(S = 0, E_1 = 0, I \neq 0, J \neq 0)$, then any $y_i - e^{x_i} \in E_2$ is either identically zero on $C(t)$ or nevee zero.

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1.2 Decomposition of $(S, E) = 0$ in K_{ex}

Since the proof of Theorem 1 uses only algebraic tools, the same statement holds in K_{ex}.

Theorem 4 For any finite set $S \subset \mathbb{Q}[\bar{x}, \bar{y}]$, the zero set of exponential system (S, E) can be written as a union of finitely many sets;

$$K^{2n}(S, E) = \bigcup_i C_i$$

Proof: Given a set S, we just proceed the decomposition of $\mathbb{C}^{2n}(S, E)$. Then transfer the decomposition onto K^{2n}. ■

1.3 Exisitential closedness and the decomposition

Definition 5 Let $(m_{ij}) \in \mathrm{M}_n(\mathbb{Z})$ be an $n \times n$-matrix. Consider a tuple of variables $(u_1, \cdots, u_n, v_1, \cdots, v_n)$. Then $(u_1, \cdots, u_n, v_1, \cdots, v_n)^{(m_{ij})}$ denotes the following tuple:

$$\left(\sum_{j=1}^{n} m_{1j} u_j, \cdots, \sum_{j=1}^{n} m_{nj} u_j, \prod_{j=1}^{n} v_j^{m_{1j}}, \cdots, \prod_{j=1}^{n} v_j^{m_{nj}} \right)$$

We call $(\overline{u}, \overline{v})^{(m_{ij})}$ an *admissible* transformation of the tuple of variables $(\overline{u}, \overline{v})$.

Recall that the strucutre K_{ex} satisfies following the existential closedness condition :

Existential closedness

Let $P_{\overline{a}}(x_1, \cdots x_n, y_1, \cdots, y_n)$ be an irreducible system of polynomials with coefficints $\overline{a}$ and $(x_1^0, \cdots, x_n^0, y_1^0, \cdots, y_n^0) \in K^{2n}$ its generic zero. If the system $P_{\overline{a}}(\overline{x}, \overline{y})$ satisfies the following conditions called *normality* and *freeness* then there is a generic zero of $P_{\overline{a}}$ such that

$$y_i^0 = \text{ex}(x_i^0), \quad i = 1, \cdots, n.$$

- (Normality condition) For any distinct $i_1, \cdots, i_m$ after any *admissible* transformation of variables, we have
$$\text{tr.deg}_{\mathbb{Q}(\overline{a})}(x_{i_1}^0, \cdots x_{i_m}^0, y_{i_1}^0, \cdots, y_{i_m}^0) \geq m$$
- (Freeness condition) After any *admissible* transformation of variables, we have for all i
$$x_i^0 \notin \text{acl}(\mathbb{Q}(\overline{a})) \text{ and } y_i^0 \notin \text{acl}(\mathbb{Q}(\overline{a}))$$

So, the exisitential closedness assures the existence of generic solutions to the exponential system (P, E) if P is normal and free.

On the other hand, the decomposition theorem describes the structure of the solution set to the exponential system if it has solutions.

1.4 PQF is not enough for an analytic Zariski structure

PQF stands for the positive quantifier free topology, i.e., the poitive quantifier free definable sets forming the basis for a topology. Since the set of positive quantifier free definable sets is closed under finite unions and finite intersections, the PQF can be introduced in K_{ex}.

Suppose we topologize K_{ex} with PQF. Then the decomposition theorem above (Theorem 4) suggests that the irreducible subsets of K^{2n} defined as the zero sets of exponential system can be viewed as *analytic* set.

However as Zilber remarks we see that PQF is not rich enough for K_{ex} being an analytic Zariski structure :

First notice that the properties of K_{ex} are all of algebraic nature; K being algebraically closed field, ex being a homomorphism from the additive structure to the multiplicative structure of the field K and the existential closedness guaranteeing the existence of solutions to exponential system.

Therefore in order to consider K_{ex} as an analytic Zariski structure, we need to say that e.g., ex$'$ = ex. For this, we need to define the derivative of ex should be *definable and analytic*.

Consider first

$$E = \{(z, x_1, x_2) \,:\, z(x_1 - x_2) = \mathrm{ex}(x_1) - \mathrm{ex}(x_2)\}$$

Then clearly $\dim E = 2$. Next put $E' = E \cap \{(z, x_1, x_2) \,:\, x_1 = x_2\}$. We see also that $\dim E' = 2$. Since E' is a subset of E of equal dimenstion, E is not irreducible if E is analytic. Therefore, by an axiom of analytic Zariski structure we have that $E = E^0 \cup E'$ for some analytic set E^0.

Consider the graph of the following function

$$g(x_1, x_2) := \begin{cases} \dfrac{\mathrm{ex}(x_1) - \mathrm{ex}(x_2)}{x_1 - x_2} & x_1 \neq x_2 \\ \mathrm{ex}(x_1) & \text{otherwise} \end{cases}$$

Notice here that $E^0 \backslash E'$ is the graph of

$$g(x_1, x_2) = \frac{\mathrm{ex}(x_1) - \mathrm{ex}(x_2)}{x_1 - \mathrm{ex}_2}, \quad x_1 \neq x_2$$

and that $E^0 \cap E'$ is the graph of the derivative of ex.

The function $g(x_1, x_2)$ above should be Zariski continuous. Although the graph of g is quantifier free definable, it is not a closed set in the PQF.

This simple example shows that the PQF is not rich enough for K_{ex} being an analytic Zariski. For details see p. 189 of [Z3].

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